

2023 年全国硕士研究生招生考试（数学二）试题及答案解析

一、选择题

1. 曲线 $y = x \ln \left(e + \frac{1}{x-1} \right)$ 的斜渐近线方程为

A. $y = x + e$.

B. $y = x + \frac{1}{e}$.

C. $y = x$.

D. $y = x - \frac{1}{e}$.

【答案】B

【解析】 $y = x \ln \left(e + \frac{1}{x-1} \right)$, $k = \lim_{x \rightarrow \infty} \frac{y}{x} = \lim_{x \rightarrow \infty} \ln \left(e + \frac{1}{x-1} \right) = \ln e = 1$

$$b = \lim_{x \rightarrow \infty} (y - x) = \lim_{x \rightarrow \infty} \left[x \ln \left(e + \frac{1}{x-1} \right) - x \right]$$

$$\text{令 } \frac{1}{x-1} = t$$

$$= \lim_{t \rightarrow 0} \left[\left(\frac{1}{t} + 1 \right) \ln(e + t) - \left(\frac{1}{t} + 1 \right) \right] = \lim_{t \rightarrow 0} \frac{(1+t) \ln(e+t) - (t+1)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{\ln(e+t) + (1+t) \cdot \frac{1}{e+t} - \frac{1}{t+1}}{1} = \ln e + \frac{1}{e} - 1 = \frac{1}{e}$$

$$y = x + \frac{1}{e}.$$

2. 函数 $f(x) = \begin{cases} \frac{1}{\sqrt{1+x^2}}, & x \leq 0, \\ (x+1)\cos x, & x > 0 \end{cases}$ 的一个原函数为

$$A. F(x) = \begin{cases} \ln(\sqrt{1+x^2} - x), & x \leq 0, \\ (x+1)\cos x - \sin x, & x > 0. \end{cases}$$

$$B. F(x) = \begin{cases} \ln(\sqrt{1+x^2} - x) + 1, & x \leq 0, \\ (x+1)\cos x - \sin x, & x > 0. \end{cases}$$

$$C. F(x) = \begin{cases} \ln(\sqrt{1+x^2} + x), & x \leq 0, \\ (x+1)\sin x + \cos x, & x > 0. \end{cases}$$

$$D. F(x) = \begin{cases} \ln(\sqrt{1+x^2} + x) + 1, & x \leq 0, \\ (x+1)\sin x + \cos x, & x > 0. \end{cases}$$

【答案】D

【解析】

$$\int (x+1)\cos x dx = \int (x+1)d\sin x = (x+1)\sin x - \int \sin x dx = (x+1)\sin x + \cos x + c$$

故排除 AB, 由于 $\lim_{x \rightarrow 0^+} F(x) = 1 \neq \lim_{x \rightarrow 0^-} F(x) = 0$, 排除 C, 故选 D.

3. 已知 $\{x_n\}, \{y_n\}$ 满足: $x_1 = y_1 = \frac{1}{2}, x_{n+1} = \sin x_n, y_{n+1} = y_n^2 (n=1, 2, \dots)$, 则当 $n \rightarrow \infty$ 时,

A. x_n 是 y_n 的高阶无穷小.

B. y_n 是 x_n 的高阶无穷小.

C. x_n 与 y_n 是等价无穷小.

D. x_n 与 y_n 是同阶但不等价的无穷小.

【答案】B

【解析】首先 $x_n > 0, x_{n+1} = \sin x_n < x_n$, 由单调有界准则可知 $\{x_n\}$ 收敛, 其次

$0 < y_n \leq \frac{1}{2}, y_{n+1} = y_n^2 < y_n$ 由单调有界准则可知 $\{y_n\}$ 也收敛, 令 $\lim_{n \rightarrow \infty} x_{n+1} = a, \lim_{n \rightarrow \infty} y_{n+1} = b$

则 $a = \sin a, b = b^2 \Rightarrow a = 0, b = 0$, 又由基础 30 讲 104 页例 7.9, 可知当 $0 < x < \frac{\pi}{2}$ 时,

$$\sin x > \frac{2x}{\pi} \text{ 且 } y_{n+1} = y_n^2 = y_n \cdot y_n \leq \frac{1}{2} \cdot y_n,$$

可得 $0 < \frac{y_{n+1}}{x_{n+1}} = \frac{y_n \cdot y_n}{\sin x_n} < \frac{\frac{1}{2} y_n}{\frac{2x_n}{\pi}} = \frac{\pi}{4} \cdot \frac{y_n}{x_n} < \left(\frac{\pi}{4}\right)^2 \frac{y_{n-1}}{x_{n-1}} < \dots < \left(\frac{\pi}{4}\right)^n \frac{y_1}{x_1} = \left(\frac{\pi}{4}\right)^n$, 又因为

$\lim_{n \rightarrow \infty} \left(\frac{\pi}{4}\right)^n = 0$, 根据夹逼准则可知 $\lim_{n \rightarrow \infty} \frac{y_{n+1}}{x_{n+1}} = 0$, 故选 B.

4. 若微分方程 $y'' + ay' + by = 0$ 的解在 $(-\infty, +\infty)$ 上有界, 则

A. $a < 0, b > 0$.

B. $a > 0, b > 0$.

C. $a = 0, b > 0$.

D. $a = 0, b < 0$.

【答案】C

【解析】当 $y'' + ay' + by = 0$ 有实根时, $a^2 - 4b \geq 0$, 设根为 r_1, r_2 , 则 $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ 或 $y = (c_1 + c_r) e^{r_1 x} (r_1 = r_2)$. 故此时不可能有解在 $(-\infty, +\infty)$ 有界. 当 $a^2 - 4b < 0$ 时, $y = (c_1 \cos \beta x + c_2 \sin \beta x) e^{ax}$, 若想解在 $(-\infty, +\infty)$ 有界, 因此 $a = 0$, 结合 $a^2 - 4b < 0$ 可得 $b > 0$. 故选 C.

5. 设函数 $y = f(x)$ 由 $\begin{cases} x = 2t + |t|, \\ y = |t| \sin t \end{cases}$ 确定, 则

A. $f(x)$ 连续, $f'(0)$ 不存在

B. $f'(0)$ 存在, $f'(x)$ 在 $x = 0$ 处不连续.

C. $f'(x)$ 连续, $f''(0)$ 不存在.

D. $f''(0)$ 存在, $f''(x)$ 在 $x = 0$ 处不连续.

【答案】C

【解析】 $\begin{cases} x = 2t + |t| \\ y = |t| \sin t \end{cases}$

当 $t \geq 0$, $\begin{cases} x = 3t \\ y = t \sin t \end{cases}$, 即 $x \geq 0$, $y = \frac{x}{3} \sin \frac{x}{3}$

当 $t < 0$, $\begin{cases} x = t \\ y = -t \sin t \end{cases}$, $x < 0$ 时 $y = -x \sin x$

$$y' = \begin{cases} \frac{1}{3} \sin \frac{x}{3} + \frac{x}{9} \cos \frac{x}{3} & x > 0 \\ 0 & x = 0 \\ -\sin x - x \cos x & x < 0 \end{cases}, \lim_{x \rightarrow 0} y'(x) = y'(0) = 0, y'(x) \text{ 在 } x=0 \text{ 处连续.}$$

$$y_+''(0) = \frac{2}{9}, y_-''(0) = -2, y''(0) \text{ 不存在.}$$

故选 C.

6. 若函数 $f(\alpha) = \int_2^{+\infty} \frac{1}{x(\ln x)^{\alpha+1}} dx$ 在 $\alpha = \alpha_0$ 处取得最小值, 则 $\alpha_0 =$

A. $-\frac{1}{\ln(\ln 2)}$ B. $-\ln(\ln 2)$ C. $\frac{1}{\ln 2}$ D. $\ln 2$

【答案】A

【解析】 $f(\alpha) = \int_2^{+\infty} \frac{1}{(\ln x)^{\alpha+1}} d(\ln x) = -\frac{1}{\alpha} (\ln x)^{-\alpha} \Big|_2^{+\infty} = \frac{1}{\alpha (\ln 2)^\alpha}$

令 $g(\alpha) = \alpha \cdot (\ln 2)^\alpha$, $g'(\alpha) = (\ln 2)^\alpha + \alpha \cdot (\ln 2)^\alpha \ln \ln 2 = 0$

$(\ln 2)^\alpha (1 + \alpha \ln \ln 2) = 0 \Rightarrow \alpha = -\frac{1}{\ln \ln 2}$, 故选 A

7. 设函数 $f(x) = (x^2 + a)e^x$, 若 $f(x)$ 没有极值点, 但曲线 $y = f(x)$ 有拐点, 则 a 的取值范围

A. $[0, 1)$ B. $[1, +\infty)$ C. $[1, 2)$ D. $[2, +\infty)$

【答案】C.

【解析】 $f'(x) = (x^2 + 2x + a)e^x$, $\Delta = 4 - 4a \leq 0 \Rightarrow a \geq 1$,

$f''(x) = (x^2 + 4x + a + 2)e^x$, $\Delta > 0 \Rightarrow 16 - 4(a + 2) > 0 \Rightarrow a < 2$, 故选 C.

8. 设 A, B 为 n 阶可逆矩阵, E 为 n 阶单位矩阵, M^* 为矩阵 M 的伴随矩阵, 则 $\begin{pmatrix} A & E \\ O & B \end{pmatrix}^* =$

A. $\begin{pmatrix} |A|B^* & -B^*A^* \\ O & |B|A^* \end{pmatrix}$

B. $\begin{pmatrix} |A|B^* & -A^*B^* \\ O & |B|A^* \end{pmatrix}$

$$C. \begin{pmatrix} |B|A^* & -B^*A^* \\ O & |A|B^* \end{pmatrix}$$

$$D. \begin{pmatrix} |B|A^* & -A^*B^* \\ O & |A|B^* \end{pmatrix}$$

【答案】D

$$\text{【解析】} \begin{bmatrix} A & E \\ O & B \end{bmatrix}^* = \begin{bmatrix} A & E \\ O & B \end{bmatrix} \cdot \begin{bmatrix} A & E \\ O & B \end{bmatrix}^{-1}$$

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} A & E \\ O & B \end{bmatrix} = \begin{bmatrix} X_1A & X_1 + X_2B \\ X_3A & X_3 + X_4B \end{bmatrix}$$

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}B^{-1} \\ O & B^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A & E \\ O & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}B^{-1} \\ O & B^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A & E \\ O & B \end{bmatrix}^* = |A| \cdot |B| \cdot \begin{bmatrix} A^{-1} & -A^{-1}B^{-1} \\ O & B^{-1} \end{bmatrix} = \begin{bmatrix} |B| \cdot A^* & -A^*B^* \\ O & |A| \cdot B^* \end{bmatrix}.$$

9. 二次型 $f(x_1, x_2, x_3) = (x_1 + x_2)^2 + (x_1 + x_3)^2 - 4(x_2 - x_3)^2$ 的规范形为

A. $y_1^2 + y_2^2$ B. $y_1^2 - y_2^2$ C. $y_1^2 + y_2^2 - 4y_3^2$ D. $y_1^2 + y_2^2 - y_3^2$

【答案】B.

$$\text{【解析】} f(x_1, x_2, x_3) = 2x_1^2 - 3x_2^2 - 3x_3^2 + 2x_1x_2 + 2x_1x_3 + 8x_2x_3$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -3 & 4 \\ 1 & 4 & -3 \end{bmatrix}$$

$$|A - \lambda E| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & -3-\lambda & 4 \\ 1 & 4 & -3-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & -3-\lambda & 4 \\ 0 & 7+\lambda & -7-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 2 \\ 1 & -3-\lambda & 1-\lambda \\ 0 & 7+\lambda & 0 \end{vmatrix}$$

$= (7+\lambda)\lambda(3-\lambda)$. 故选 B.

10. 已知向量 $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \beta_1 = \begin{pmatrix} 2 \\ 5 \\ 9 \end{pmatrix}, \beta_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. 若 γ 既可由 α_1, α_2 线性表示, 也可由

β_1, β_2 线性表示, 则 $\gamma =$

A. $k \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, k \in \mathbf{R}$ B. $k \begin{pmatrix} 3 \\ 5 \\ 10 \end{pmatrix}, k \in \mathbf{R}$ C. $k \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, k \in \mathbf{R}$ D. $k \begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix}, k \in \mathbf{R}$

【答案】D

【解析】

$$\gamma = k_1 \alpha_1 + k_2 \alpha_2 = l_1 \beta_1 + l_2 \beta_2, \quad k_1 \alpha_1 + k_2 \alpha_2 - l_1 \beta_1 - l_2 \beta_2 = 0,$$

$$\begin{cases} x_1 = k_1 \\ x_2 = k_2 \\ x_3 = -l_1 \\ x_4 = -l_2 \end{cases} \quad x_1 \alpha_1 + x_2 \alpha_2 + x_3 \alpha_3 + x_4 \alpha_4 = 0$$

$$\begin{cases} x_1 = 3k \\ x_2 = -k \end{cases}, \gamma = 3k \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - k \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = k \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}.$$

二、填空题

11. 当 $x \rightarrow 0$ 时, 函数 $f(x) = ax + bx^2 + \ln(1+x)$ 与 $g(x) = e^{x^2} - \cos x$ 是等价无穷小, 则 $ab =$ _____.

【答案】-2

【解析】 $\lim_{x \rightarrow 0} \frac{ax + bx^2 + \ln(1+x)}{e^{x^2} - \cos x} = 1$

$$\lim_{x \rightarrow 0} \frac{ax + bx^2 + \left(x - \frac{1}{2}x^2\right)}{1 + x^2 - \left(1 - \frac{1}{2}x^2\right)} = 1$$

$$\Rightarrow (a+1) = 0 \quad b - \frac{1}{2} = \frac{3}{2} \Rightarrow a = -1, b = 2 \Rightarrow ab = -2.$$

12. 曲线 $y = \int_{-\sqrt{3}}^x \sqrt{3-t^2} dt$ 的弧长为_____.

【答案】 $\sqrt{3} + \frac{4}{3}\pi$

【解析】

$$y = \int_{-\sqrt{3}}^x \sqrt{3-t^2} dt, y' = \sqrt{3-x^2}, \text{ 由 } 3-x^2 \geq 0, x \in [-\sqrt{3}, \sqrt{3}] = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{1+3-x^2} dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-x^2} dx = \left[\frac{1}{2} x \sqrt{4-x^2} + \frac{4}{2} \arcsin \frac{x}{2} \right]_{-\sqrt{3}}^{\sqrt{3}} = \sqrt{3} + \frac{4}{3} \pi.$$

13. 设函数 $z = z(x, y)$ 由 $e^z + xz = 2x - y$ 确定, 则 $\frac{\partial^2 z}{\partial x^2} \Big|_{(1,1)} = \underline{\hspace{2cm}}.$

【答案】 $-\frac{3}{2}$

【解析】 利用隐函数公式法可知 $\frac{\partial z}{\partial x} = -\frac{z-2}{e^z+x}$, 当 $x=1, y=1 \Rightarrow z=0$, 则

$$\frac{\partial z}{\partial x} \Big|_{z=0, x=1} = -\frac{z-2}{e^z+x} \Big|_{z=0, x=1} = 1,$$

$$\text{可得 } \frac{\partial^2 z}{\partial x^2} \Big|_{x=1} = -\frac{\frac{\partial z}{\partial x}(e^z+x) - \left(e^z \frac{\partial z}{\partial x} + 1\right)(z-2)}{(e^z+x)^2} \Big|_{x=1} = -\frac{3}{2}.$$

14. 曲线 $3x^3 = y^5 + 2y^3$ 在 $x=1$ 对应点处的法线斜率为 $\underline{\hspace{2cm}}.$

【答案】 $-\frac{11}{9}$

【解析】

利用隐函数公式法可知 $\frac{dy}{dx} \Big|_{x=1, y=1} = -\frac{9x^2}{-5y^4 - 6y^2} \Big|_{x=1, y=1} = \frac{9}{11}$, 则法线斜率为 $-\frac{11}{9}$.

15. 设连续函数 $f(x)$ 满足: $f(x+2) - f(x) = x, \int_0^2 f(x) dx = 0$, 则 $\int_1^3 f(x) dx = \underline{\hspace{2cm}}.$

【答案】 $\frac{1}{2}$

【解析】

$$\int_1^3 f(x) dx = \int_1^0 f(x) dx + \int_0^2 f(x) dx + \int_2^3 f(x) dx, \text{ 由于 } \int_0^2 f(x) dx = 0$$

所以原式为 $-\int_0^1 f(x) dx + \int_2^3 f(x) dx$, 由于 $\int_2^3 f(x) dx = \int_0^1 f(x+2) dx$, 故原式

$$= \int_0^1 [f(x+2) - f(x)] dx = \int_0^1 x dx = \frac{1}{2}.$$

16. 已知线性方程组
$$\begin{cases} ax_1 + x_3 = 1, \\ x_1 + ax_2 + x_3 = 0, \\ x_1 + 2x_2 + ax_3 = 0, \\ ax_1 + bx_2 = 2 \end{cases}$$
 有解, 其中 a, b 为常数, 若 $\begin{vmatrix} a & 0 & 1 \\ 1 & a & 1 \\ 1 & 2 & a \end{vmatrix} = 4$, 则 $\begin{vmatrix} 1 & a & 1 \\ 2 & a \\ a & b & 0 \end{vmatrix} =$

【答案】8

【解析】

$$\begin{vmatrix} 1 & a & 1 \\ 1 & 2 & a \\ a & b & 0 \end{vmatrix} = 4 \neq 0, r \begin{vmatrix} a & 0 & 1 \\ 1 & a & 1 \\ 1 & 2 & a \\ a & b & 0 \end{vmatrix} = 3, r \begin{vmatrix} a & 0 & 1 & 1 \\ 1 & a & 1 & 0 \\ 1 & 2 & a & 0 \\ a & b & 0 & 2 \end{vmatrix} = 3,$$

$$\begin{vmatrix} a & 0 & 1 & 1 \\ 1 & a & 1 & 0 \\ 1 & 2 & a & 0 \\ a & b & 0 & 2 \end{vmatrix} = 0, 1 \begin{vmatrix} 1 & a & 1 \\ 1 & 2 & a \\ a & b & 0 \end{vmatrix} - 2 \begin{vmatrix} a & 0 & 1 \\ 1 & a & 1 \\ 1 & 2 & a \end{vmatrix} = 0, \begin{vmatrix} 1 & a & 1 \\ 1 & 2 & a \\ a & b & 0 \end{vmatrix} = 8$$

三、解答题

17. 设曲线 $L: y = y(x) (x > e)$ 经过点 $(e^2, 0)$, L 上任一点 $P(x, y)$ 到 y 轴的距离等于该点处的切线在 y 轴上的截距.

(1) 求 $y(x)$;

(2) 在 L 上求一点, 使该点处的切线与两坐标轴所围三角形的面积最小, 并求此最小面积.

【解析】

由题意得 $y = y'(x - x) + y$ 为切线方程, 切线在 y 轴上得截距为 $-x \cdot y' + y$

$$\text{则 } x = -x \cdot y' + y \Rightarrow y' - \frac{y}{x} = -1.$$

$$y(x) = e^{\int \frac{1}{x} dx} \left[\int + e^{\int -\frac{1}{x} dx} dx + c \right]$$

$$= x \left[\int \frac{1}{x} dx + c \right]$$

$$= x(-\ln x + c)$$

又 $x=1$, $y=2$ 则 $c=2$ 因此 $y(x)=x(-\ln x+2)$

$$(2) f'(x) = y(x) = x(-\ln x + 2) = 0$$

则 $x=0$ 或 $x=e^2$.

又 $x>0$ 故 $f(x)$ 的驻点为 $x=e^2$

$$f''(x) = -\ln x + 2 + x \cdot \left(-\frac{1}{x}\right)$$

$$f'(e^2) = -2 + 2 - 1 = -1 < 0$$

$$\text{故 } f(e^2) \text{ 为最大值, } \int_1^{e^2} x(-\ln x + 2) dx = \frac{e^4 - 5}{4}$$

18. 求函数 $f(x, y) = xe^{\cos y} + \frac{x^2}{2}$ 的极值.

【解析】

$$\begin{cases} f'_x = e^{\cos y} + x = 0 \\ f'_y = ke^{\cos y}(-\sin y) = 0 \end{cases}, \text{ 得驻点 } (-e, 2n\pi), \left(-\frac{1}{e}, (2n+1)\pi\right);$$

$$f''_{xx} = 1$$

$$f''_{xy} = e^{\cos y}(-\sin y)$$

$$f''_{yy} = xe^{\cos y} \sin^2 y + ke^{\cos y}(-\cos y)$$

对于 $(-e, 2n\pi)$, $A=1, B=0, C=e^2, AC-B^2>0, A>0$. 有极小值 $f(-e, 2n\pi) = -\frac{e^2}{2}$

对于 $\left(-\frac{1}{e}, (2n+1)\pi\right)$, $A=1, B=0, C=-\frac{1}{e^2}, AC-B^2<0$, 无极值.

19. 已知平面区域 $D = \{(x, y) | 0 \leq y \leq \frac{1}{x\sqrt{1+x^2}}, x \geq 1\}$.

(1)求 D 的面积;

(2)求 D 绕 x 轴旋转所成旋转体的体积.

【解析】

$$(1) \int_1^{+\infty} \frac{1}{x\sqrt{1+x^2}} dx \stackrel{x=\tan t}{=} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\tan t \cdot \sec t} \cdot \sec^2 t dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sec t}{\tan t} dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc t dt = \ln(\sqrt{2}+1)$$

$$(2) \int_1^{+\infty} \pi \left(\frac{1}{x\sqrt{1+x^2}} \right)^2 dx = \int_1^{+\infty} \pi \frac{1}{x^2(1+x^2)} dx = \int_1^{+\infty} \pi \left(\frac{1}{x^2} - \frac{1}{1+x^2} \right) dx = \pi \left(1 - \frac{\pi}{4} \right)$$

20.(12分)设平面有界区域 D 位于第一象限, 由曲线 $x^2+y^2-xy=1$, $x^2+y^2-xy=2$ 与直线 $y=\sqrt{3}x$, $y=0$ 围成, 计算 $\iint_D \frac{1}{3x^2+y^2} dx dy$.

【解析】

$$\iint_D \frac{1}{3x^2+y^2} dx dy$$

$$= \int_0^{\frac{\pi}{3}} d\theta \int_{\frac{1}{\sqrt{1-\cos\theta\sin\theta}}}^{\frac{2}{\sqrt{1-\cos\theta\sin\theta}}} \frac{1}{r^2+2r^2\cos^2\theta} \cdot r dr$$

$$= \pi \frac{\sqrt{3} \ln 2}{24}$$

21. (12分) 设函数 $f(x)$ 在 $[-a, a]$ 上具有 2 阶连续导数, 证明:

(1) 若 $f(0)=0$, 则存在 $\xi \in (-a, a)$, 使得 $f''(\xi) = \frac{1}{a^2}[f(a)+f(-a)]$;

(2) 若 $f(x)$ 在 $(-a, a)$ 内取得极值, 则存在 $\eta \in (-a, a)$ 使得

$$|f''(\eta)| \geq \frac{1}{2a^2} |f(a) - f(-a)|.$$

【解析】

$$(1) f(x) = f(0) + f'(0)x + \frac{1}{2}f''(\xi)x^2$$

则 $f(a) = f'(0)a + \frac{1}{2}f''(\xi_2)a^2$, $f(-a) = f'(0)(-a) + \frac{1}{2}f''(\xi_1)a^2$, 其中 $\xi_1 \in (-a, 0)$,

$$\xi_2 \in (0, a).$$

$$f(-a) + f(a) = \frac{1}{2} [f''(\varepsilon_1) + f''(\varepsilon_2)] a^2$$

$$\text{由介值定理可知平均值 } \frac{1}{2} [f''(\varepsilon_1) + f''(\varepsilon_2)] = \frac{f(-a) + f(a)}{a^2} = f''(\xi), \xi \in [\xi_1, \xi_2] \subset (-a, a),$$

∴ 即证

(2)

设 $f(x)$ 在 $x=x_0$ 处取得极值 即 $x_0 \in (-a, a), f'(x_0) = 0$

$$\therefore f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)^2$$

代入 $x = -a, x = a$

$$f(-a) = f(x_0) + \frac{f''(\eta_1)}{2}(a + x_0)^2 \quad (1), \eta_1 \in (-a, x_0)$$

$$f(a) = f(x_0) + \frac{f''(\eta_2)}{2}(a - x_0)^2 \quad (2), \eta_2 \in (x_0, a)$$

(2) - (1) 得

$$f(a) - f(-a) = \frac{f''(\eta_2)}{2}(a - x_0)^2 - \frac{f''(\eta_1)}{2}(a + x_0)^2$$

$$|f(a) - f(-a)| = \left| \frac{f''(\eta_2)}{2}(a - x_0)^2 - \frac{f''(\eta_1)}{2}(a + x_0)^2 \right|$$

$$\leq \left| \frac{f''(\eta)}{2}(a - x_0)^2 \right| + \left| \frac{f''(\eta)}{2}(a + x_0)^2 \right|$$

$$\leq \left| \frac{f''(\eta)}{2} \right| [(a - x_0)^2 + (a + x_0)^2]$$

$$= \left(\frac{f''(\eta)}{2} \right) (2a^2 + 2x_0^2)$$

$$= |f''(\eta)| (a^2 + x_0^2)$$

$$\leq |f''(\eta)| \cdot 2a^2, \quad \text{其中 } f''(\eta) = \max \{f''(\eta_1) \cdot f''(\eta_2)\}, \eta \in (-a, a)$$

$$\therefore |f''(\eta)| \geq \frac{1}{2a^2} |f(a) - f(-a)|.$$

22. 设矩阵 A 满足对任意 x_1, x_2, x_3 均有 $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ 2x_1 - x_2 + x_3 \\ x_2 - x_3 \end{pmatrix}.$

(1) 求 A ;

(2) 求可逆矩阵 P 与对角矩阵 Λ , 使得 $P^{-1}AP = \Lambda$.

【解析】

解(1) 由题可知, $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\therefore A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

$$(2) |A - \lambda E| = -(2 + \lambda)(\lambda - 2)(\lambda + 1) = 0$$

$$\therefore A \text{ 中 } \lambda_1 = 2, \quad \lambda_2 = -1, \quad \lambda_3 = -2$$

$$A \text{ 中 } \lambda_1 \text{ 对应的线性无关特征向量 } \alpha_1 = (4, 3, 1)^T.$$

$$A \text{ 中 } \lambda_2 \text{ 对应的线性无关特征向量 } \alpha_2 = \left(-\frac{1}{2}, 0, 1\right)^T$$

$$A \text{ 中 } \lambda_3 \text{ 对应的线性无关特征向量 } \alpha_3 = (0, -1, 1)^T$$

$$\therefore P = (\alpha_1, \alpha_2, \alpha_3)$$

$$P^{-1}AP = \begin{pmatrix} 2 & & \\ & -1 & \\ & & -2 \end{pmatrix}$$

